Hall Algebra

Thursday, March 25, 2021 12:18 AM

- Let $\vec{Q} = (I, H)$, I = vertices, H = edges
- let $k = \mathbb{F}_q$, and assume $\vec{Q}$ has no loops

Def

$$\mathrm{Rep}_k \vec{Q} = \{(\underline{V}, \underline{x}) \mid \underline{V} = \bigoplus_{i \in I} V_i, \underline{x} = (x_h)\}$$

- $s, t : H \to I$ source, target of edge
- $x_h : V_{s(h)} \to V_{t(h)}$
- $\underline{\dim} \underline{V} = (\dim V_i)_{i \in I}$

Def Let $\vec{Q}$ be a quiver, $q = p^r$, p prime

$$H(\vec{Q}, q) = \text{free } \mathbb{Z}[q^{\pm 1}] \text{ module w/ basis } [\mathrm{Rep}_k \vec{Q}]$$

$$[\underline{M_1}] * [\underline{M_2}] = \sum_{[L]} |F^{L}_{M_1, M_2}| [L]$$

$$F^{L}_{M_1, M_2} = \{\underline{X} \subset \underline{L} \mid \underline{X} \simeq \underline{M_2}, \underline{L}/\underline{X} \simeq \underline{M_1}\}$$

Ex: $\vec{Q} = \bullet \Rightarrow \mathrm{Rep}_k \vec{Q} = \mathbb{F}_q\text{-mod}$

Let $S = \overset{k}{\bullet}$, $nS = \overset{k^{\oplus n}}{\bullet}$

Claim: $[nS] * [S] = [n+1]_q [(n+1)S]$

Pf: B/c $\mathbb{F}_q$-mod is s.s, only one possible $L = (n+1)S$, then note

$$F^{(n+1)S}_{nS, S} = \{X \subset k^{n+1} \mid X \simeq k\} = \mathbb{P}^n(\mathbb{F}_q)$$

$$\Rightarrow |F^{(n+1)S}_{nS,S}| = |\mathbb{P}^n(\mathbb{F}_q)| = [n+1]_q$$

Cor: $[S]^{*n} = [n]_q! \, [nS]$

Thm (Ringel): Let $\mathfrak{g}_{\vec{Q}}$ be Kac-Moody lie alg associated to $\vec{Q}$, and let $\mathfrak{n}^+_{\vec{Q}}$ = positive part of $\mathfrak{g}_{\vec{Q}}$. Then we have an embedding of algs

$$U_q(\mathfrak{n}^+_{\vec{Q}}) \hookrightarrow H(\vec{Q}, q^2)$$

When $\vec{Q}$ is a Dynkin quiver, this map is an isomorphism

Constructible Hall Algebra 1

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Rem (1) $\vec{Q} \rightsquigarrow$ sym matrix $C_{\vec{Q}} = 2I - A_{\vec{Q}}$ ← adjacency matrix

(2) can extend to $U_q'(b_Q^+) \hookrightarrow \tilde{A}(Q, q^2)$

(3) $U_q'(\mathfrak{g}) = U_q(\mathfrak{g})$ − "derivation" when $\mathfrak{g}$ = affine lie alg
= what appears in [Nak]

- Not clear how to categorify $H(\vec{Q}, q)$
- So will define iso alg that is "more geometric"

Def Fix $\alpha \in \mathbb{N}^I$

$$E_\alpha(q) = \bigoplus_{(i \to j) \in H} \mathrm{Hom}(k^{\alpha_i}, k^{\alpha_j})$$

= "all rep of $\vec{Q}$ w/ $\underline{\dim}\, V = \alpha$"

$$G_\alpha = \prod_i GL_{\alpha_i}(k)$$

- $\underline{g} \in G_\alpha \curvearrowright \underline{y} = (y_h)_{h \in H} \in E_\alpha$ by conjugation

$$g \cdot \underline{y} = (g_{t(h)} y_h g_{s(h)}^{-1})_{h \in H}$$

Def Fix $\alpha \in \mathbb{N}^I$

$$H_\alpha^{con}(\vec{Q}, q) = \mathrm{Fun}(E_\alpha(\mathbb{F}_q), \mathbb{Z}[q^{\pm 1}])^{G_\alpha}$$

Def Let

$$E_{\alpha,\beta}(q) = \left\{ (\underline{x}, \underline{W}) \;\middle|\; \begin{array}{l} \cdot\ \underline{x} \in E_{\alpha+\beta} \\ \cdot\ \underline{W} \subset k^{\alpha+\beta},\ \underline{\dim}\, \underline{W} = \beta \\ \cdot\ \underline{x}\, \underline{W} \subset \underline{W} \end{array} \right\}$$

$$E^{(1)}_{\alpha,\beta}(q) = \left\{ (\underline{y}, \underline{W}, \psi_\alpha, \psi_\beta) \;\middle|\; \begin{array}{l} \cdot\ (\underline{y}, \underline{W}) \in E_{\alpha,\beta} \\ \cdot\ \psi_\alpha : k^{\alpha+\beta}/\underline{W} \xrightarrow{\sim} k^\alpha \\ \cdot\ \psi_\beta : \underline{W} \xrightarrow{\sim} k^\beta \end{array} \right\}$$

$$\begin{array}{ccccc} E^{(1)}_{\alpha\beta}(q) & \xrightarrow{r} & E_{\alpha\beta}(q) & & \\ \downarrow p & & & \searrow \pi & \\ E_\alpha(q) \times E_\beta(q) & & & & E_{\alpha+\beta}(q) \end{array}$$

- r, π are forgetting maps
- $p(\underline{y}, \underline{W}, \psi_\alpha, \psi_\beta) = \left(\psi_\alpha(\underline{y}|_{k^{\alpha+\beta}/\underline{W}}), \psi_\beta(\underline{y}|_{\underline{W}})\right)$

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$G_\alpha \times G_\beta \curvearrowright E^{(1)}_{\alpha,\beta}$ by

$$(g_\alpha, g_\beta)\cdot(\underline{V}, \underline{W}, \psi_\alpha, \psi_\beta) = (\underline{V}, \underline{W}, g_\alpha\psi_\alpha, g_\beta\psi_\beta)$$

$G_{\alpha+\beta} \curvearrowright E_\alpha \times E_\beta$ trivially, $G_{\alpha+\beta} \curvearrowright E^{(1)}_{\alpha,\beta}$ by

$$g\cdot(\underline{V}, \underline{W}, \psi_\alpha, \psi_\beta) = (g\underline{V}g^{-1}, g\underline{W}, \psi_\alpha g^{-1}, \psi_\beta g^{-1})$$

__Facts:__ (1) p is smooth and $G_{\alpha+\beta}\times G_\alpha \times G_\beta$-equi

(2) r is a principal $G_\alpha \times G_\beta$ bundle over $E_{\alpha,\beta}$

(3) π is proper and $G_{\alpha+\beta}$-equi

Mult in $H^{con}(\vec{Q}, q) = \bigoplus_{\alpha \in \mathbb{N}^I} H^{con}_\alpha(\vec{Q}, q)$

$$\star \quad H^{con}_\alpha(\vec{Q}, q) \times H^{con}_\beta(\vec{Q}, q) \to H^{con}_{\alpha+\beta}(\vec{Q}, q)$$

$$\star \quad \mathrm{Fun}(E_\alpha)^{G_\alpha} \times \mathrm{Fun}(E_\beta)^{G_\beta} \longrightarrow \mathrm{Fun}(E_{\alpha+\beta})^{G_{\alpha+\beta}}$$

$$(f_1, f_2) \longmapsto (\pi)_! (r^*)^{-1} p^*(f_1 \times f_2)$$

__Def__ Given $\varphi: X \to Y$,

$$\varphi^*: \mathrm{Fun}(Y) \to \mathrm{Fun}(X)$$
$$f \longmapsto f\circ\varphi$$

$$\varphi_!: \mathrm{Fun}(X) \longrightarrow \mathrm{Fun}(Y)$$
$$f \longmapsto (\varphi_! f)(y) = \sum_{z \in \varphi^{-1}(y)} f(z)$$

- B/c r is a principal $G_\alpha \times G_\beta$ bundle $\Rightarrow$ have equivalence

$$(r^*)^{-1}: \mathrm{Fun}(E^{(1)}_{\alpha,\beta})^{G_{\alpha+\beta}\times G_\alpha\times G_\beta} = \mathrm{Fun}(E_{\alpha,\beta})^{G_{\alpha+\beta}} \quad (4)$$

$$f \longmapsto (r^*)^{-1}(f)(\underline{x}, \underline{w}) := f(\underline{x}, \underline{w}, \psi_\alpha, \psi_\beta)$$

any iso

$\Rightarrow$ Final formula is

$$(f_1 \star f_2)(\underline{x}) = \sum_{\substack{W \text{ s.t.} \\ \underline{x}W \subset W}} f_1\left(\psi_\alpha\left(\underline{x}|_{k^{\alpha+\beta}/W}\right)\right) f_2\left(\psi_\beta\left(\underline{x}|_W\right)\right)$$

- (1), (3) $\Rightarrow \pi_!, p^*$ preserve equivariance

$(r^*)^{-1}$ changes equi, so final product is in

$\mathrm{Fun}(E_{\alpha+\beta})^{G_{\alpha+\beta}}$ as desired

Categorification 1

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Thrm: Have iso of alg

$$H(\vec{Q}, q) \cong H^{con}(\vec{Q}, q)$$

- RHS has a path to categorification via sheaves - functions dictionary of Grothendieck

  F constructible sheaf $\longmapsto$ $\mathrm{tr}\,F$ constructible function

- It turns out we will need perverse sheaves

Goal: (1) Given quiver $\vec{Q}$, categorify $H^{con}(\vec{Q}, q)$

- Replace finite set $E_\alpha(\mathbb{F}_q)$ w/ scheme $E_\alpha / \overline{\mathbb{F}_q}$

$$E_\alpha = \prod_{(i \to j) \in H} \mathrm{Hom}(\mathbb{A}^{\alpha_i}, \mathbb{A}^{\alpha_j}) = \prod_{(i \to j) \in H} \mathbb{A}^{\alpha_i \times \alpha_j}$$

- By dictionary should replace

$$\mathrm{Fun}(E_\alpha(\mathbb{F}_q))^{G_\alpha} \rightsquigarrow D^b_{c, G_\alpha}(E_\alpha)$$

Warning: $D^b_{c,G}(X) \neq D^b_c(X) +$ "equivariant condition" but it's true for perverse sheaves!

Def/Thrm: A G-equivariant perverse sheaf is the datum $(F^\bullet, \phi)$, $\phi: a^* F^\bullet \simeq p_2^* F^\bullet$ + cocycle + identity axioms

Miracle: If G is connected, then forgetful functor

$$\mathrm{For}_G : P_G(X) \longrightarrow P(X)$$

is fully faithful with essential image all $F^\bullet$ s.t. $\exists\, \phi : a^* F^\bullet \simeq p_2^* F^\bullet$ (cocycle + identity auto!)

$\Rightarrow$ Equivariant perverse sheaves is a property not a structure when G connected (e.g. think of V instead of (V, ρ))

Lem: $IC(Y, \mathcal{L})$ is G-equivariant $\iff$ Y is G-stable, $\mathcal{L}$ is G-equivariant local system

Categorification 2

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(2) Categorify mult in $H^{con}(\vec{Q}, q)$

$$\begin{array}{ccccc} & E^{(1)}_{\alpha,\beta} & \xrightarrow{\ r\ } & E_{\alpha,\beta} & \\ p & \downarrow & & & \searrow \pi \\ & E_\alpha \times E_\beta & & & E_{\alpha+\beta} \end{array}$$

$\star : D^b_{c,G_\alpha}(E_\alpha) \times D^b_{c,G_\beta}(E_\beta) \longrightarrow D^b_{c,G_{\alpha+\beta}}(E_{\alpha+\beta})$

$(F_1^\bullet, F_2^\bullet) \longmapsto \pi_!(r^*)^{-1}p^*(F_1^\bullet \boxtimes F_2^\bullet)$

Rem: We have using following categorifications

$\times \longrightarrow \boxtimes$

(4) $\longrightarrow D^b_{c,G_{\alpha+\beta}\times G_\alpha \times G_\beta}(E^{(1)}_{\alpha,\beta}) \xrightarrow{\sim} D^b_{c,G_{\alpha+\beta}}(E_{\alpha+\beta})$

Warning: There is a cohomological shift $[\dim E_{\alpha,\beta}]$ in $\star$ that I ignored

Obs: $F^\bullet \star (G_1^\bullet \oplus G_2^\bullet) = (F^\bullet \star G_1^\bullet) \oplus (F^\bullet \star G_2^\bullet)$

Lem: $\mathbb{D}(F^\bullet \star G^\bullet) \simeq \mathbb{D}(F^\bullet) \star \mathbb{D}(G^\bullet)$

pf: Use π proper, p smooth, and for $F^\bullet, G^\bullet$ constructible

$$\mathbb{D}(F^\bullet \boxtimes G^\bullet) = \mathbb{D}(F^\bullet) \boxtimes \mathbb{D}(G^\bullet)$$

(3) Categorify $U_q(n^+_{\mathbb{Q}}) \hookrightarrow H^{con}(\vec{Q}, q)$

Def: Let $e_i = (0, \dots, 1, \dots, 0) \in \mathbb{N}^I$. B/c $\vec{Q}$ has no loops, $E_{e_i} = \{s_i\} \simeq pt$. Let

$$\mathbb{1}_{e_i} := \underline{\mathbb{C}}_{E_{e_i}}$$

Rem: B/c pt is smooth of dim 0, $\mathbb{1}_{e_i}$ is a perverse sheaf on E_{e_i}, in fact we have

$$\mathbb{1}_{e_i} = IC(E_{e_i}, \underline{\mathbb{C}}) = IC_{[e_i]}$$

Def Lusztig' sheaves are

$$\mathbb{L} = \langle \mathbb{1}_{\alpha_1} \star \cdots \star \mathbb{1}_{\alpha_n} \mid \alpha_i = e_{k_i} \text{ for some } k_i \in I \rangle$$

Hall Category

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Def $\underline{H}_\gamma \subseteq D^b_{c,G_\gamma}(E_\gamma)^{ss}$ defined by

$$\underline{H}_\gamma = \langle \mathbb{1}_{\alpha_1} \star \cdots \star \mathbb{1}_{\alpha_n} \in \underline{L} \rangle_{\oplus, [1], \underline{\oplus}}$$

$\underline{H}_{\vec{Q}} = \coprod_{\gamma \in \mathbb{N}^I} \underline{H}_\gamma$ is the Hall category

Rem: $\underline{H}_{\vec{Q}}$ closed under $\star$ b/c of obs, so this will give $K_{\vec{Q}} = K_\oplus(\underline{H}_{\vec{Q}})$ the structure of an algebra.

Rem: Let $L_{\alpha_1,\dots,\alpha_n} = \mathbb{1}_{\alpha_1} \star \cdots \star \mathbb{1}_{\alpha_n}$, and

$$E_{\alpha_1,\dots,\alpha_n} = \{(\underline{V}, \underline{k}^{\alpha_1+\dots+\alpha_n} = \underline{W_1} \supset \underline{W_2} \supset \cdots \supset \underline{W_n})\}$$

where $\underline{V} \in E_{\alpha_1+\dots+\alpha_n}$, $\underline{W}_i$ $\underline{V}$ stable, and $\underline{\dim} \frac{W_k}{W_{k+1}} = \alpha_k$

Let $\tau: E_{\alpha_1,\dots,\alpha_n} \to E_{\alpha_1+\dots+\alpha_n}$. Then

$$L_{\alpha_1,\dots,\alpha_n} = \tau_!(\underline{\mathbb{C}}[\dim E_{\alpha_1,\dots,\alpha_n}])$$

(4) Categorify the Relations

Def $[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + \dots + q^{-(n-1)} \in \mathbb{N}[q, q^{-1}]$

Def Let $P \in$ triangulated cat w/ shift $[1]$

If $R(q) = \sum c_i q^i \in \mathbb{N}[q, q^{-1}]$, set

$$[R(q)]P = \bigoplus_i P^{\oplus c_i}[i]$$

Lem: Let $d_1, \dots, d_r \in \mathbb{Z}^+$, $d = d_1 + \dots + d_r$. Let $P_{d_1,\dots,d_r} \subset GL_d(k)$ be associated parabolic subgrp

$$\mathcal{B}_{d_1,\dots,d_r} = \frac{GL_d(k)}{P_{d_1,\dots,d_r}}.$$

$$\Rightarrow \sum_i \dim H^i(\mathcal{B}_{d_1,\dots,d_r}[\dim \mathcal{B}_{d_1,\dots,d_r}]) q^i = \frac{[d]!}{[d_1]! \cdots [d_r]!}$$

Fundamental Relations

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Example 1: $\vec{Q} = \bullet$, $(\mathbb{1}_{e_1})^n$ will be

$$\mathbb{1}_{e_1} \ast \cdots \ast \mathbb{1}_{e_1} = L_{1,\dots,1} = \pi_!(\mathbb{C}[\dim E_{1,\dots,1}])$$

No edges $\Rightarrow \underline{V} = 0$ so

$$E_{1,\dots,1} = \{0, \bar{k}^{\oplus n} = W_1 \supset \dots \supset W_n \mid \dim W_i/W_{i+1} = 1\}$$

$$= Gl_n/B \leftarrow \text{compact}$$

while $E_{1+\dots+1} = E_n = \{\bar{k}^n\} = pt \Rightarrow \pi_! = \pi_*$

$=$ take cohomology

$$\Rightarrow L_{1,\dots,1} = \bigoplus_k H^k(Gl_n/B, \mathbb{C})[\dim Gl_n/B]$$

$$\mathbb{1}_{e_1}^n = [n]! \, \mathbb{C}_{E_n} = [n]! \, \mathbb{1}_{ne_1}$$

Ex 2: $\vec{Q} = \bullet \quad \bullet$, $e_1 = (1,0)$ $e_2 = (0,1)$

Claim: $\mathbb{1}_{e_1} \ast \mathbb{1}_{e_2} = \mathbb{1}_{e_2} \ast \mathbb{1}_{e_1}$

Again, no edges $\Rightarrow \underline{V} = 0$

$$E_{e_1,e_2} = \{\underline{0}, \bar{k}^{(1,1)} \supset \underline{W_2} \mid \dim \underline{W_2} = (0,1)\}$$

$$= \{\overset{0}{\bullet} \quad \overset{\bar{k}}{\bullet}\} = pt = \{\overset{\bar{k}}{\bullet} \quad \overset{0}{\bullet}\} = E_{e_2,e_1}$$

$$\Rightarrow \mathbb{1}_{e_1} \ast \mathbb{1}_{e_2} = \pi_!(\mathbb{C}) = \mathbb{1}_{e_2} \ast \mathbb{1}_{e_1}$$

Ex 3: $\vec{Q} = \bullet \rightarrow \bullet$, $e_1 = (1,0), e_2 = (0,1)$

$L_{e_1,e_2,e_1} = ?$ $L_{e_1,e_1,e_2} = ?$, $L_{e_2,e_1,e_1} = ?$

All are s.s. complexes on

$$E_{2e_1+e_2} = \{\overset{\bar{k}^2}{\bullet} \rightarrow \overset{\bar{k}}{\bullet}\} \simeq \mathbb{A}^2$$

orbits of $G_{2e_1+e_2}$:

$O_1 =$ rk 1 matrices $\simeq \mathbb{A}^2 \setminus (0,0)$

$O_0 =$ rk 0 matrices $\simeq (0,0)$

Can check: Table of stalks

$IC(O_1) =$

	-2	-1	0
O_1	1	0	0
O_0	1	0	0

$IC(O_0) =$

	-2	-1	0
O_1	0	0	0
O_0	0	0	1

Fundamental Relations 2

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- can check $\dim E_{e_1,e_2,e_1} = 2$. Thus $E_{e_1,e_2,e_1} \xrightarrow{\pi_{121}} E_{2e_1+e_2}$

$$L_{e_1,e_2,e_1} = (\pi_{121})_*(\underline{\mathbb{C}}[2])$$

(i) Compute fibers of π_{121}

Let $x_1 \in O_1$, $x_0 \in O_0$, then

	$\pi^{-1}(x_i)$
O_1	pt
O_0	$\mathbb{P}^1$

(ii) Use PBC to fill in table of stalks

$$TOS((\pi_{121})_*(\underline{\mathbb{C}}[2])) \overset{PBC}{=} \begin{array}{c|c} O_1 & H^*(\pi^{-1}(x_1), \mathbb{C}) \\ \hline O_0 & H^*(\pi^{-1}(x_0), \mathbb{C}) \end{array}$$

=

	-2	1	0
O_1	1	0	0
O_0	1	0	1

(iii) Use Decomposition theorem to show iso

Notice $TOS((\pi_{121})_*(\underline{\mathbb{C}}[2]))$

$= TOS(IC(O_1)) + TOS(IC(O_0))$

Decomp $\Rightarrow$ $(\pi_{121})_*(\underline{\mathbb{C}}[2])$ is ss complex

$\Rightarrow$ det by TOS

$\Rightarrow$ $(\pi_{121})_*(\underline{\mathbb{C}}[2]) \cong IC(O_1) \oplus IC(O_0)$

We can repeat to obtain

$$L_{e_1,e_1,e_2} = IC(O_1)[1] \oplus IC(O_1)[-1]$$

$$L_{e_2,e_1,e_1} = IC(O_0)[1] \oplus IC(O_0)[-1]$$

$$= L_{e_1,e_2,e_1}[1] \oplus L_{e_1,e_2,e_1}[-1]$$

$\Rightarrow$ $L_{e_1,e_1,e_2} \oplus L_{e_2,e_1,e_1} \cong [2]\, L_{e_1,e_2,e_1}$

This Categorifies Serre relation •—•